

Cromlech, menhirs and celestial sphere: an unusual representation of the Lorentz group

Jerzy Kocik

Department of Mathematics
Southern Illinois University, Carbondale, IL62901
jkocik@siu.edu

Abstract

We present a novel representation of the Lorentz group, the geometric version of which uses “reversions” of a sphere while the algebraic version uses pseudo-unitary 2×2 matrices over complex numbers and quaternions, and Clifford algebras in general. A remarkably simple formula for relativistic composition of velocities and an accompanying geometric construction follow. The method is derived from the diffeomorphisms of the celestial sphere induced by Lorentz boost.

Keywords: $SU(1, 1)$, inversive geometry, reversions, complex numbers, quaternions, Clifford algebra, relativity, space-time, composition of velocities, Thomas rotation, celestial sphere, golden ratio, visualization, menhir.

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1. Introduction and results

In one dimension, the formula for relativistic composition of velocities has a simple algebraic form:

$$v \oplus w = \frac{v + w}{1 + vw}, \quad (1.1)$$

discovered in 1904 by Henri Poincaré [10]. The above operation turns the open interval $(-1, 1)$ into an Abelian group. It may be visualized in terms of a simple geometric gadget reproduced in Figure 1.1 from [3, 4]. Once we go beyond collinearity of velocities, the algebraic simplicity of the Poincaré formula is lost. This is because the boosts, which are among the generators of the Lorentz Lie group $SO(1, n)$, do not form a subgroup and a rotational component emerges in a product of two boosts.

In the present article, we propose an approach which has two expressions: algebraic and geometric. The basic idea is to present velocities in a reduced form called *menhirs*, the construction of which is shown in Figure 1.3. The two formalisms are:

- **Geometric representation** — composition of boosts obtains a very simple form in terms of geometric constructions based on inversions (“reversions”) of a sphere through the points called menhirs.

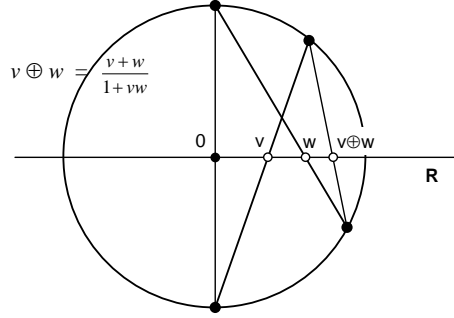


FIGURE 1.1: Visualization of the Poincaré formula (1.1) for velocities as points v and w on real line \mathbb{R} . The velocity of relativistic composition $v \oplus w$ is determined by the quadrilateral inscribed in the unit circle as shown.

- **Algebraic representation** — In the algebraic form, we use division algebras like \mathbb{R} , \mathbb{C} or \mathbb{H} and other Clifford algebras. Remarkably, the “menhir representation” of velocities restores the Poincaré-like simplicity of the formula for the composition of boosts.

Summary: We present the material as a “tail of three groups” by taking a trip through the diagram of homomorphisms and relations presented in Figure 1.2. The fraction-like symbol $\frac{G}{X} \Downarrow$ indicates that the group G acts on the set X . The three columns are the three wings of the construction: (1) relativity and Lorentz group, (2) the geometry of sphere, and (3) its algebraic version in terms of certain matrix groups. Here is the general legend to this diagram.

1. The left top corner denotes the Lorentz group acting on the Minkowski vector space $M \cong \mathbb{R}^{1,n}$. This action may be restricted to the isotropic vectors, the light cone $\mathbb{R}_0^{1,n}$ (map “ \subset ”). This induces an action on *rays* in the light cone, elements of the subset $\mathbb{P}\mathbb{R}_0^{1,n}$ of the projective space. Topologically it is a sphere, $\mathbb{P}\mathbb{R}_0^{1,n} \cong S^{n-1}$, called the **celestial sphere**. The Lorentz group can be perceived via deformations of the celestial sphere, the **aberration** of its points.
2. An observer understood as a space-like subspace of $\mathbb{R}^{1,n}$ may identify the celestial sphere with the unit sphere S^{n-1} (called here **cromlech**) in her space. We show that every velocity may be represented by a point $p \in D \subset \mathbb{R}^n$ called **menhir** inside the cromlech via a certain “menhir map” and show how this point determines the aberration in a simple purely geometric way.
3. The last column refers to the matrix representation of the reversion sphere with the use of a field \mathbb{F} . In the case of (1+2)-Minkowski space $n = 2$, $\mathbb{F} = \mathbb{C}$ and the group is the unitary group $SU(1, 1)$ acting on $S^1 \subset \mathbb{C}$ via the fractional linear maps. The case of $n = 4$ involves quaternions, $\mathbb{F} = \mathbb{H}$, the corresponding group is $SU_{1,1}(\mathbb{H}) \equiv Sp(1, 1)$, and the celestial sphere is S^3 . This wing leads to a simple algebraic formula for addition of velocities. A generalization to other Clifford algebras takes care of $n > 4$.

“star” through the center and then through the menhir.

B. Algebraic version of menhir calculus. The geometry of menhirs can be given algebraic form in terms of division algebras $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (real numbers, complex numbers and quaternions) which take care of 1-, 2-, and 4-dimensional space. Both the velocity and its menhir become elements of \mathbb{F} inside the unit disk $D \subset \mathbb{F}$. We shall use Greek letters when the menhirs are understood as algebra elements, i.e., $\varepsilon \in \mathbb{F}$ for \mathbf{e} . Now, the map

$$\mu : D \rightarrow D : v \mapsto \varepsilon = \mu(v)$$

defined geometrically in Figure 1.3 translates into relations:

$$v = \frac{2\varepsilon}{1 + |\varepsilon|^2}, \quad \varepsilon = \frac{v}{1 + \sqrt{1 - |v|^2}}, \quad (1.3)$$

where $|\varepsilon|^2 = \varepsilon \bar{\varepsilon}$. A composition of two boosts, described by menhirs ε_1 and ε_2 , is equivalent to a single boost described by menhir

$$\varepsilon_1 \boxplus \varepsilon_2 = \frac{\varepsilon_1 + \varepsilon_2}{1 + \bar{\varepsilon}_1 \varepsilon_2} \quad (1.4)$$

(which—quite remarkably—has the same form as the Poincaré formula (1.1)), followed by a (Thomas) rotation represented by

$$\rho = \frac{1 + \varepsilon_2 \bar{\varepsilon}_1}{1 + \bar{\varepsilon}_2 \varepsilon_1} \quad (1.5)$$

These two equations may be read off from a single matrix “master equation”

$$\begin{bmatrix} 1 & \varepsilon_2 \\ \bar{\varepsilon}_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \varepsilon_1 \\ \bar{\varepsilon}_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon_2 \bar{\varepsilon}_1 & 0 \\ 0 & 1 + \bar{\varepsilon}_2 \varepsilon_1 \end{bmatrix} \begin{bmatrix} 1 & \varepsilon_1 \boxplus \varepsilon_2 \\ (\varepsilon_1 \boxplus \varepsilon_2)^* & 1 \end{bmatrix} \quad (1.6)$$

where the matrices on the left represent the boosts and on the right a boost composed with rotation. The order of products is here essential due to non-commutativity of quaternions.

The situation for complex numbers is represented in the following commutative diagram:

$$\begin{array}{ccccc} \text{menhirs:} & \mathbb{C} \times \mathbb{C} & \xrightarrow{\boxplus} & \mathbb{C} & \\ & \downarrow \mu \times \mu & & \downarrow \mu & \\ \text{velocities:} & \mathbb{C} \times \mathbb{C} & \xrightarrow{\oplus} & \mathbb{C} & \end{array} \quad (1.7)$$

or, $\mu(a \boxplus b) = \mu(a) \oplus \mu(b)$.

To clarify, the matrices in (1.6) form a **group**, namely pseudo-unitary group $\text{SU}(1, 1)$ over one of the division algebras, homomorphic to the corresponding Lorentz group. However, operation \boxplus is a non-associative and defines on the unit disk a **loop** (D, \boxplus) , a quasigroup with an identity. The operation of composition of velocities is an isomorphic loop (D, \oplus) .

Algebraic conclusion: An interesting correspondence emerges between the normed division algebras and the Lorentz groups:

$$\begin{array}{llll}
 \mathbb{R} & \text{SO}_0(1, 1) & \leftrightarrow & \text{SU}_{1,1}(\mathbb{R}) \cong \mathbb{R} \\
 \mathbb{C} & \text{SO}_0(1, 2) & \leftrightarrow & \text{SU}_{1,1}(\mathbb{C}) \equiv \text{SU}(1, 1) \\
 \mathbb{H} & \text{SO}_0(1, 4) & \leftrightarrow & \text{SU}_{1,1}(\mathbb{H}) \equiv \text{Sp}(1, 1) \\
 \mathbb{O} & \text{SO}_0(1, 8) & \leftrightarrow & \text{SU}_{1,1}(\mathbb{O})
 \end{array}
 \quad \left| \quad
 \begin{array}{ll}
 \text{SO}_0(1, 2) \leftrightarrow \text{SL}(n, \mathbb{R}) \\
 \text{SO}_0(1, 3) \leftrightarrow \text{SL}(n, \mathbb{C}) \\
 \text{SO}_0(1, 5) \leftrightarrow \text{SL}(n, \mathbb{H}) \\
 \text{SO}_0(1, 9) \leftrightarrow \text{SL}(n, \mathbb{O})
 \end{array}
 \right.$$

The first three cases, real, complex and quaternionic ($\mathbb{R}, \mathbb{C}, \mathbb{H}$), are considered in the following sections, while the octonions, \mathbb{O} , will be analyzed elsewhere. This association is alternative to relation $\text{SL}(n, \mathbb{F}) \leftrightarrow \text{SO}(1, n+1)$ shown above on the right side [12], often recalled in the context of supersymmetry [6, 1].

Example: To see the simplicity of the “menhir calculus” consider two orthogonal velocities and the corresponding menhirs given as complex numbers:

$$v_1 = \frac{4}{5}, \quad v_2 = \frac{3i}{5} \quad \Rightarrow \quad \varepsilon_1 = \frac{1}{2}, \quad \varepsilon_2 = \frac{i}{3}$$

Using (1.4), one finds the that the composition of the two implied boosts corresponds the menhir $\varepsilon = \varepsilon_1 \boxplus \varepsilon_2$ and velocity $v = v_1 \oplus v_2$:

$$\varepsilon = \frac{1/2 + i/3}{1 + 1/2 \cdot i/3} = \frac{20 + 9i}{37} \quad \Rightarrow \quad v = \frac{2\varepsilon}{1 + \bar{\varepsilon}\varepsilon} = \frac{4}{5} + \frac{9}{25}i$$

Thus velocity’s direction is that of $5 + 3i$, and the speed is $4\sqrt{34}/25 \approx 14/15$. The rotational part of the composition of the boosts is

$$\rho = \frac{1 + \frac{1}{2}\frac{i}{3}}{1 - \frac{1}{2}\frac{i}{3}} = \frac{6 + i}{6 - i} = \frac{35 + 12i}{37}$$

which indicates that the angle of rotation to be $\theta = \arccos(35/37) \approx 19^\circ$.

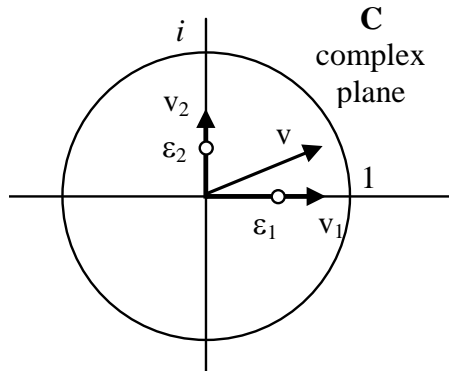


FIGURE 1.5: Example. The “sum” of velocities does not coincide with the sum of vectors.

* * *

An alternative connection between the spinors and the Lorentz group was introduced in [2] and popularized in [9]. For analysis of relativistic composition of velocities in terms of a grupoid see [8].

2. Reversions and unitary groups

Here we review the concept of reversion groups following [4]. Let $D \subset E \cong \mathbb{R}^n$ be a unit disk and $K = \partial D$ be the unit sphere in the Euclidean space E .

Definition 2.1. A **reversion** of a sphere K through a point $p \in D$ is a map

$$\mathbf{p} : K \rightarrow K : A \mapsto A\mathbf{p}$$

such that points $(p, A, A\mathbf{p})$ are collinear and $A \neq A\mathbf{p}$. See Figure 2.1 left for illustration.

Reversions may be composed. In particular, $\mathbf{p}^2 = \text{Id}$. They generate a group that we shall denote

$$\text{Rev}(n) = \text{gen} \{ \mathbf{p} \mid p \in D \}$$

where $n = \dim D$. The subgroup of elements that consists of the composition of an even number of reversions will be denoted by $\text{Rev}_o(n) \subset \text{Rev}(n)$.

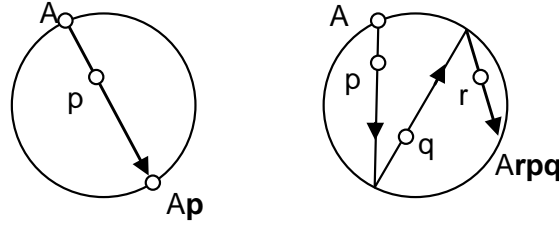


FIGURE 2.1: Definition of reversion (left) and a composition of reversions (right)

In the special case of $n = 2$, we may employ complex numbers and represent D and K as unit disk and circle at the origin, $D = \{ \varepsilon \in \mathbb{C} \mid |\varepsilon| < 1 \}$ and $K = \{ z \in \mathbb{C} \mid |\varepsilon| = 1 \}$.

Proposition 2.2. A reversion through point represented by a complex number $\varepsilon \in \mathbb{C}$ can be then represented by a linear fractional transformation

$$z \rightarrow z' = \frac{z - \varepsilon}{\bar{\varepsilon}z - 1} = \begin{bmatrix} 1 & -\varepsilon \\ \bar{\varepsilon} & -1 \end{bmatrix} \cdot z \quad (2.1)$$

where the matrix on the right side is the standard notation for such maps.

One simply checks that this map preserves K and that the image z' is collinear with ε and z by verifying that $(z\varepsilon)(z' - \varepsilon) \in \mathbb{R}$.

As an example of an even-order reversion, consider the following useful case:

Corollary 2.3. The composition of reversion through the origin followed by reversion through point $\varepsilon \in D$ has the following form

$$z \rightarrow \frac{(-z) - \varepsilon}{\bar{\varepsilon}(-z) - 1} = \frac{z + \varepsilon}{\bar{\varepsilon}z + 1} = \begin{bmatrix} 1 & \varepsilon \\ \bar{\varepsilon} & 1 \end{bmatrix} \cdot z. \quad (2.2)$$

Group characterization of the matrices. The (2.1) and (2.2) suggest that the groups of reversions may be represented by matrix groups.

Proposition 2.4. *The groups of reversions in dimension 2 are isomorphic to the special pseudo-unitary groups, namely:*

$$\begin{aligned}\text{Rev}(2) &\cong \text{PSU}^\pm(1, 1) \\ \text{Rev}_o(2) &\cong \text{PSU}(1, 1)\end{aligned}\tag{2.3}$$

Proof: The pseudo-unitary group $U(1, 1)$ consists of matrices that preserve the inner product given by

$$g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in the sense that $A^*gA = g$. One can identify two special subgroups:

$$\begin{aligned}\text{SU}^\pm(1, 1) &= \{ A \in U(1, 1) \mid \det A = \pm 1 \} \\ \text{SU}(1, 1) &= \{ A \in U(1, 1) \mid \det A = 1 \}\end{aligned}\tag{2.4}$$

Clearly, $\text{SU}(1, 1) \subset \text{SU}^\pm(1, 1)$. In the case of $\det A = 1$ the matrices are of type:

$$A = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}, \quad |a|^2 - |b|^2 = 1, \quad a, b \in \mathbb{C}$$

Under projectivization this group becomes $\text{PSU}(1, 1)$. However it is very useful to use matrices that are scaled so that the connection with geometry (and later relativity) is simple. Hence we start with the the group $\mathbb{R}_+ \times \text{SU}(1, 1)$, which consists of matrices

$$A = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}, \quad |a|^2 - |b|^2 > 0, \quad a, b \in \mathbb{C}$$

Note that the matrices of (2.2) are of this type. Under projectivization, the groups are identified, $P(\mathbb{R}_+ \times \text{SU}(1, 1)) \cong \text{PSU}(1, 1)$. The same goes for odd-order reversions for which the matrices are of form

$$\begin{bmatrix} a & -b \\ \bar{b} & -\bar{a} \end{bmatrix} = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with similar rules for a and b . \square

We end with noting that every element of our even version of the group may be decomposed:

$$\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} \begin{bmatrix} 1 & b/a \\ (b/a)^* & 1 \end{bmatrix}$$

(Note that $a \neq 0$ since $|a|^2 > 0$).

This establishes the isomorphism of the bottom right and central part of diagram in Figure 1.2 for the 2-dimensional case and complex numbers. A similar correspondence for $\dim D = 4$ and quaternions will be discussed in Section 5.

3. As above, so below – Stonehenge applied

Notation:

1. $\mathbf{M} \equiv \mathbb{R}^{1,n}$ — Minkowski vector space with an inner product G of signature $(1, n)$.
2. $\Lambda \equiv \text{SO}_0(1, n)$ — **Lorentz group**, the connected component of the symmetry group $\text{SO}(1, n)$ of M .
3. \mathbf{M}_F — future unit hyperboloid consisting of the future-oriented unit vectors; a single sheet from the 2-component set $\{\mathbf{w} \in \mathbb{R}^{1,n} \mid |\mathbf{w}|^2 = 1\}$.
4. $\mathbb{R}_0^{1,n}$ — the **light cone**, the subset of isotropic vectors of $\mathbb{R}^{1,n}$, $G(w, w) = 0$.
5. $\text{PR}_0^{1,n}$ — the **celestial sphere**, the projective space $\text{PR}^{1,n}$ restricted to the light cone. Topologically, $\text{PR}_0^{1,n} \equiv S^{n-1}$.
6. **Tempus** — a unit future-oriented time-like vector $\mathbf{T} \in \mathbf{M}_F$. It determines an **observer**, that is the split $\mathbf{M} \cong (\text{span } \mathbf{T}) \oplus \mathbf{T}^\perp$ where $\mathbf{T}^\perp = \{\mathbf{w} \in M \mid \mathbf{w} \perp \mathbf{T}\}$ is the associated perpendicular subspace (instantaneous space).
7. **Lab** — an n -dimensional Euclidean space \mathbf{E} together with a linear injective isometry $\lambda : \mathbf{E} \rightarrow \mathbf{M}$. The embedding determines $\mathbf{T}_\lambda \in \mathbf{M}_F$ such that $\mathbf{T}_\lambda \perp \lambda(\mathbf{E})$.

Making sense of “adding velocities”. A lot of conceptual trouble may be avoided by introducing a notion of a “lab”, namely a reference Euclidean space $\mathbf{E} \cong \mathbb{R}^n$ together with injective isometry to the Minkowski vector space

$$\lambda : \mathbf{E} \rightarrow \mathbf{M} \quad (3.1)$$

One may think of a flat “planetoid” like one in Figure 3.1 as its intuitive metaphor. Suppose we can control the behavior of this planetoid: we may rotate it or give it a boost or in general, send the image of \mathbf{E} to a new orientation in \mathbf{M} , so that λ is replaced by a new embedding $\lambda' = g \circ \lambda$, which is a composition of the original λ with an element of the group $g \in \Lambda$. In particular, a boost by velocity $\mathbf{v} \in \mathbf{E}$ is understood as a map $B_{\lambda, \mathbf{v}} \in \Lambda$, for simplicity denoted $B_{\mathbf{v}}$, such that the new “tempus” admits decomposition $\mathbf{T}_{\lambda'} = \mathbf{T}_\lambda + \lambda(\mathbf{v})$, and no rotation appears, i.e., $(\mathbf{T}_\lambda \wedge \mathbf{T}_{\lambda'})^\perp$ remains fixed under the boost.

We will make sense of adding velocities “ $\mathbf{v} \oplus \mathbf{w}$ ” in the space \mathbf{E} by the means of **planning**. Given an ordered pair of vectors, $\mathbf{v}, \mathbf{w} \in \mathbf{E}$ we execute two boosts: first $B_{\mathbf{v}}$ by velocity vector \mathbf{v} and then $B_{\mathbf{w}} = B_{\mathbf{w}, B_{\mathbf{v}} \circ \lambda}$. (Note that vector \mathbf{w} is the *same* vector in \mathbf{E} but *different* vector in space-time, namely $B_{\mathbf{v}}(\lambda(\mathbf{w})) \in \mathbf{M}$.)

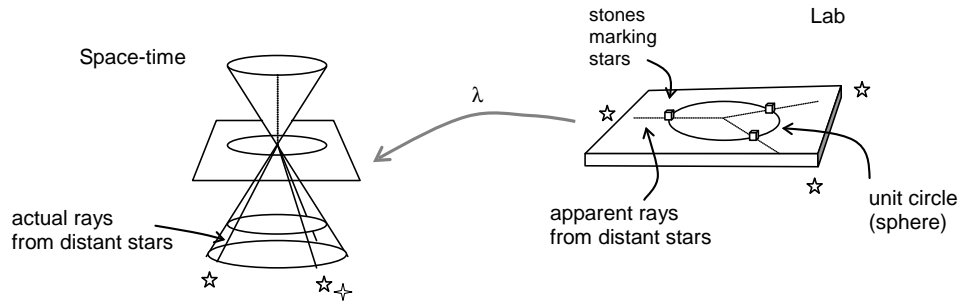


FIGURE 3.1: Building a relativistic “Stonehenge”.

One can reach the same result by a single boost by a velocity denoted $v \oplus w \in E$ followed by a certain rotation denoted $R_{v,w}$:

$$B_w \circ B_v = R_{v,w} \circ B_{v \oplus w} \quad (3.2)$$

The velocity $v \oplus w$ is viewed as the “composition of velocities” and it is only a part of the story. We are interested in finding the formulas for both components, the effective vector $v \oplus w$ and the angle and axis of rotation $R_{v,w}$.

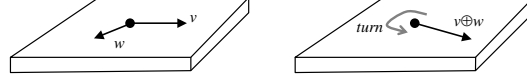


FIGURE 3.2: Preplanned two boosts are marked on the ground of planetoid (lab). They are equivalent to a single boost followed by rotation

Cromlech. When we look at a starry sky at night – we see a *celestial sphere* made by rays from distant stars. Mathematically, we see points of the projective space restricted to the directions in the null cone, $\mathbb{PR}_0^{1,n}$ (topologically equivalent to sphere S^{n-1}).

To follow our 2-dimensional example of planetoid \mathbb{R}^2 , imagine that a sky-watcher draws a unit circle $K \subset \mathbf{E}$, called **cromlech**. Then he sets stones on this circle to mark particularly interesting stars (theoretically – all points) as seen from the center on the horizon. He would perceive boosts and rotations as aberration of star positions, that is as conformal diffeomorphisms of the celestial sphere, or, equivalently, as diffeomorphisms of K . Clearly, it generalizes to any dimension.

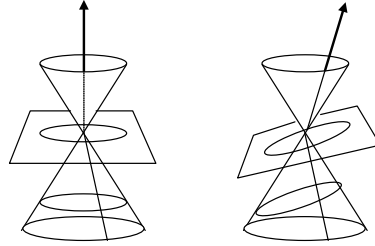


FIGURE 3.3: “Celestial sphere” as a sphere in space

Here is a more precise description: For a particular observer (tempus) $\mathbf{T} = \mathbf{T}_\lambda$, we can map the celestial sphere to the unit sphere in an instantaneous space \mathbf{T}^\perp by cutting the light-cone by the hyperplane parallel to space \mathbf{T}^\perp at the level $(-\mathbf{T})$ and then projecting this intersection on the space \mathbf{T}^\perp along \mathbf{T} (see Figure 3.3). Mathematically, we get an $(n-1)$ -sphere in $\lambda(\mathbf{E})$:

$$S^{n-1} \cong (\mathbf{T}^\perp - \mathbf{T}) \cap \mathbb{R}_0^{1,n} + \mathbf{T}$$

The map identifying K with the celestial sphere is thus

$$\pi_\lambda : K \rightarrow \mathbb{R}_0^{1,n} : s \mapsto \text{span} \{ \lambda(s) - \mathbf{T}_\lambda \}$$

Let us now look how the boost-induced diffeomorphism of K may be viewed geometrically. Given velocity as a vector (point) $\mathbf{v} \in E$ inside the cromlech, $|\mathbf{v}|^2 < 1$, the associated boost $B_{\mathbf{v}}$ will make the stars on the sky undergo an **aberration**, they will shift towards the boost direction, except the star in front and behind that remain unaffected.



FIGURE 3.4: **Left:** A single boost makes a shift of stars in the direction of the velocity.
Right: Shift of stars in cromlech due to a boost measured along the velocity axis.

Here is our initial simple observation:

Proposition 3.1. *The stars represented in our the cromlech will change the positions so that their projections x on the axis along the velocity is transformed to a new value x' according to*

$$x \rightarrow x' = \frac{x + v}{1 + vx} \quad (3.3)$$

Proof: Denote $s = \sinh \omega$, $c = \cosh \omega$, $t = \tanh \omega = v$. The transformation of the stars on the celestial sphere corresponds to action of A^{-1} on the vectors $[t = -1, x]$, where

$$A = \begin{bmatrix} c & s \\ s & c \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} c & -s \\ -s & c \end{bmatrix}.$$

Calculations may be done for the two-dimensional subspace span by x and t :

$$\begin{bmatrix} c & -s \\ -s & c \end{bmatrix} \begin{bmatrix} -1 \\ x \end{bmatrix} = \begin{bmatrix} -c - sx \\ s + cx \end{bmatrix} \doteq \begin{bmatrix} -1 \\ \frac{s+cx}{s+sx} \end{bmatrix} \Rightarrow x' = \frac{cx + s}{sx + c}$$

Now, divide the numerator and denominator by $\cosh \theta$ and use the fact that $s/c = \tanh \theta = v$ to get Equation (3.3).

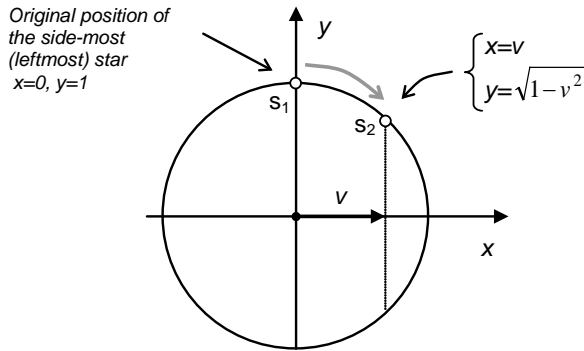


FIGURE 3.5: Velocity of boost and the shift of the side-most stars in 2-dimensional case

Corollary 3.2: The two antipodal star stones that did not change their positions determine the axis of velocity v . The side-most stars will move to position such that the corresponding stones will be aligned with the velocity stone. See Figure 3.5.

Proof: Substitute $x = 0$ for the side-most stars to (3.3), and $x = \pm 1$ for the axis points. \square

4. Celestial sphere, menhirs and star motion

Now we show how to construct the deformation of the celestial sphere in a purely geometric method. Let us go back to the “planetoid” setup: consider a lab $\mathbf{E} \cong \mathbb{R}^n$ embedded in the Minkowski space $\mathbf{M} \cong \mathbb{R}^{1,n}$. Denote $D \subset \mathbf{E}$ the disk of vectors of the norm not exceeding 1. They will represent velocities, $|\mathbf{v}| < 1$, candidates for boosts. The unit sphere $K = \partial D$ will be called “cromlech” for the reasons explained. Its center is denoted O .

Definition 4.1. A **menhir** associated to velocity \mathbf{v} is a vector $\mathbf{e} \in D$ the construction of which is shown in Figure 4.1. The corresponding one-to-one map

$$\mu : D \ni \mathbf{v} \longrightarrow \mathbf{e} \in D$$

will be called menhir map. Vectors \mathbf{e} will geometrically interpreted as a point in D .

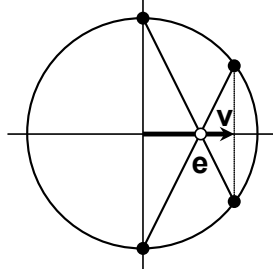


FIGURE 4.1: Position of menhir \mathbf{e} for a given velocity

Proposition 4.2. *The relation between velocity \mathbf{v} and the associated menhir $\mathbf{e} = \mu(\mathbf{v})$ is*

$$\mathbf{v} = \frac{2\mathbf{e}}{1 + e^2} \quad (4.1)$$

where $e = |\mathbf{e}|$.

Proof: Use similarity of triangles, Figure 4.1, to get $e : 1 = v : (1 + \sqrt{1 - v^2})$. Solve for v . \square

Remark: The relation between the v and e (absolute values) can be presented in the following elegant form

$$\frac{1 - v}{1 + v} = \left(\frac{1 - e}{1 + e} \right)^2 \quad (4.2)$$

When solved for \mathbf{v} , it resolves in (4.1), when resolved for \mathbf{e} , one gets the inverse relation

$$\mathbf{e} = \frac{\mathbf{v}}{1 + \sqrt{1 - v^2}}. \quad (4.3)$$

Just for completeness, here are some other forms of these relations

$$v = \frac{(1 + e)^2 - (1 - e)^2}{(1 + e)^2 + (1 - e)^2} \quad e = \frac{\sqrt{1 + v} - \sqrt{1 - v}}{\sqrt{1 + v} + \sqrt{1 - v}} \quad (4.4)$$

Now we can state our first important result.

Theorem 4.3. *A boost by velocity \mathbf{v} causes deformation of the celestial sphere K in the way that coincides with a composition of two reversion: through the origin followed and the menhir $\mathbf{e} = \mu(\mathbf{v})$. That is, a star at position $A \in K$ will shift to position*

$$A \mapsto A' = Aoe$$

Proof: Pick a point $s_1 \in K$ and construct its image s_2 via the stated composition of reversion $\mathbf{e}o$, Figure 4.2. Project the points on the axis defined by the velocity \mathbf{v} , interpreted as a number axis. The two shaded triangles are similar and by Thales' theorem,

$$\sqrt{1 - x'^2} : (x' - e) = \sqrt{1 - (-x)^2} : (e + x).$$

After squaring both sides and organizing as a polynomial equation, notice that one may factor out a term $(x + x')$. The remaining terms are linear in x' and we readily get

$$x' = \frac{x + \frac{2e}{1+e^2}}{1 + x \frac{2e}{1+e^2}}.$$

This, comparing with (4.1) and (3.3), gives the result. \square

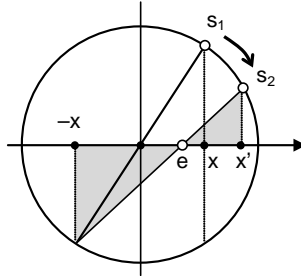


FIGURE 4.2: Similar triangles – proof of Theorem 4.3

We have a simple geometric method of describing the action of the Lorentz group on the celestial sphere. represented by the cromlech — see Figure 4.3.

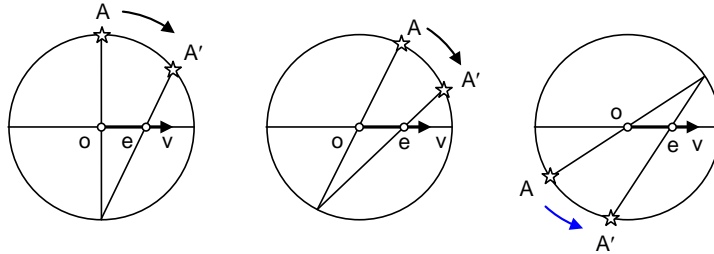


FIGURE 4.3: Shift of stars constructed with a menhir

Towards geometric method of composition of velocities. Now we see how two boosts may be represented via deformation of the celestial sphere $K \rightarrow K$: Suppose the first performed boost is along velocity \mathbf{v} , followed by the boost along \mathbf{w} . Let the corresponding menhirs are e and f , $\mu(e) = \mathbf{v}$ and $\mu(f) = \mathbf{w}$. Then the star shift is

$$A \mapsto A' = Aoeof$$

This will be simplified to a form that indicates the “sum of velocities” $v \oplus w$ more directly in Section 6.

Remark (Phi in the sky): The relation between the velocity and the position of the menhir e a non-uniform monotone map, shown in Figure 4.4 (for absolute values). The end-points coincide: the null velocity $v = 0$ corresponds to $e = 0$ and $v = 1$ to $e = v = 1$. The rule of thumb is that for the small values we have $e \approx v/2$.

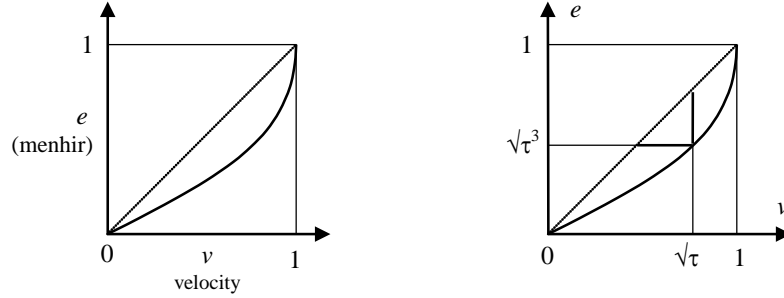


FIGURE 4.4: Position of the menhir as a function of velocity

One may ask: For what velocity the discrepancy between v and e is greatest. A quick excursion in calculus reveals that it happens for

$$v = \varphi^{-1/2} \quad \text{where} \quad \varphi = (1 + \sqrt{5})/2$$

is the golden ratio! This implies that in the situation of the greatest discrepancy the menhir cuts the segment representing the velocity in the golden proportion. Indeed, denoting the “golden cut” by $\tau = 1/\varphi = (\sqrt{5} - 1)/2$, we get velocity $v = \sqrt{\tau}$ and the menhirs position $e = \sqrt{\tau}^3$, and therefore $v : e = \varphi$.

$$\begin{array}{ccc}
 \text{GEOMETRY} & & \text{PHYSICS} \\
 \text{reversions} & & \text{relativity} \\
 \frac{\text{Rev}_0(n)}{S^{n-1}} \downarrow & \xleftrightarrow{1:1} & \frac{\text{SO}_0(1,n)}{\mathbb{R}_0^{1,n}} \downarrow \\
 \downarrow \subset & & \downarrow \subset \\
 \frac{\text{Rev}(n)}{S^{n-1}} \downarrow & \xleftrightarrow{1:1} & \frac{\text{SO}_0^+(1,n)}{\mathbb{R}_0^{1,n}} \downarrow
 \end{array} \tag{4.5}$$

5. Algebraic version of the “menhir calculus”

In this section we derive algebraic representation of the geometry developed in the previous section. We will consider in succession three division algebras $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, which describe 1, 2, 4- dimensional space (the standard physical 3D space is contained in \mathbb{H}). Menhirs, as points inside the disk represent indirectly velocities via the map (from menhirs to velocities):

$$\mu : D^n \rightarrow D^n : v \rightarrow \varepsilon$$

defined by relation

$$v = \frac{2\varepsilon}{1 + |\varepsilon|^2} \quad v, \varepsilon \in \mathbb{F} \quad (5.1)$$

A. One-dimensional case and \mathbb{R}

Recall that collinear velocities add up according to the Poincaré relativistic formula:

$$v \oplus w = \frac{v + w}{1 + vw} \quad (5.2)$$

This scalar algebraic formula turns the segment $D = (-1, 1)$ into an Abelian group (the inverse of v is $-v$, the neutral element is 0). We may translate this to the algebraic expression for behavior of the menhir values of e . Result is somewhat unexpected:

Theorem 5.1. *The menhirs corresponding to collinear velocities obey the same addition formula as the vectors of velocity, i.e., they “add up” à la Poincaré:*

$$e_1 \boxplus e_2 = \frac{e_1 + e_2}{1 + e_1 e_2} \quad (5.3)$$

That is, we have a group isomorphism $\mu : (D, \oplus) \rightarrow (D, \boxplus)$:

$$\mu(v) \boxplus \mu(w) = \mu(v \oplus w)$$

Proof: Substitute (5.1) to (5.2) and simplify

$$\begin{aligned} v_1 \oplus v_2 &= \frac{v_1 + v_2}{1 + v_1 v_2} = \frac{\frac{2e_1}{1+e_1^2} + \frac{2e_2}{1+e_2^2}}{1 + \frac{2e_1 2e_2}{(1+e_1^2)(1+e_2^2)}} \\ &= \frac{2(e_1 + e_2)(1 + e_1 e_2)}{(e_1 + e_2)^2 + (1 + e_1 e_2)^2} \quad (\text{rearranging terms}) \\ &= \frac{2 \frac{e_1 + e_2}{1 + e_1 e_2}}{\left(\frac{e_1 + e_2}{1 + e_1 e_2}\right)^2 + 1} \quad (\text{factor } (1 + e_1 e_2)^2) \end{aligned}$$

Comparing this with (5.2) gives the result. \square

Remark: Note that the map μ is the square in the sense of the group action, $\mu(e) = e \boxplus e$. In other words, the menhirs are the “group square roots” of velocities (or rather “halves” – if you prefer the additive terminology). Note that due to the commutativity and associativity in this 1-dimensional case we have

$$(e \boxplus e) \boxplus (f \boxplus f) = (e \boxplus f) \boxplus (e \boxplus f)$$

B. Two-dimensions and complex numbers \mathbb{C}

Two-dimensional case is sufficient to illustrate composition of a pair of non-collinear velocities. We equip the space \mathbf{E} (the ground of our 2-dimensional planetoid) with a complex structure, $\mathbb{C} \cong \mathbb{R}^2$. The choice of the real axis is inessential. The cromlech is now $K = \{z \in \mathbb{C} : |z|^2 = 1\}$.

Theorem 5.2. *Under a boost, the points on the unit circle (stones of cromlech representing stars) are transformed via complex linear fractional maps*

$$z \rightarrow z' = \frac{z + \varepsilon}{\bar{\varepsilon}z + 1} \equiv \begin{bmatrix} 1 & \varepsilon \\ \bar{\varepsilon} & 1 \end{bmatrix} \cdot z \quad (5.4)$$

where ε is the complex number representing the menhir (reduced velocity) and $z \in K$ is a unit complex number. In particular, the map preserves K .

Proof: This follows directly from Corollary 2.3. \square

Remark 1: Rotation is realized by $z \rightarrow e^{i\varphi}z$. Thus boost given by ε and followed by a rotation is of form

$$z \rightarrow z'' = e^{i\varphi} \cdot \begin{bmatrix} 1 & \varepsilon \\ \bar{\varepsilon} & 1 \end{bmatrix} \cdot z = e^{i\varphi} \frac{z + \varepsilon}{\bar{\varepsilon}z + 1}$$

Remark 2: The product of a scalar and a matrix behaves in the following way:

$$a \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \equiv \begin{bmatrix} a\alpha & a\beta \\ \gamma & \delta \end{bmatrix} \equiv \begin{bmatrix} \alpha & \beta \\ \gamma/a & \delta/a \end{bmatrix}. \quad (5.5)$$

(to reflect the meaning of the matrices as Möbius action).

Now, we are ready to see how a composition of boosts is represented in stone (menhir) realization in Argand plane. Here is the result:

Theorem 5.3. [Master equation – complex version]. *Let $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ be two complex numbers that represent menhirs for two vectors $v_1, v_2 \in \mathbb{C}$. The composition $B(v_2)B(v_1)$ of two pre-designed boosts is equivalent to a single boost $B(v_1 \oplus v_2)$ followed by a rotation, where the menhir representation of $v \oplus w$ is*

$$\varepsilon_1 \boxplus \varepsilon_2 = \frac{\varepsilon_1 + \varepsilon_2}{1 + \bar{\varepsilon}_1 \varepsilon_2} \quad (5.6)$$

and the associated Thomas rotation φ is twice the $\text{Arg}(1 + \bar{\varepsilon}_1 \varepsilon_2)$, or, in the form $\rho = e^{i\varphi}$, it is

$$\rho = e^{i\varphi} = \frac{1 + \bar{\varepsilon}_1 \varepsilon_2}{1 + \varepsilon_1 \bar{\varepsilon}_2} \quad (5.7)$$

This can be gathered in this \mathbf{C} -version of the “master equation”:

$$\begin{bmatrix} 1 & \varepsilon_2 \\ \bar{\varepsilon}_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \varepsilon_1 \\ \bar{\varepsilon}_1 & 1 \end{bmatrix} = \frac{1 + \bar{\varepsilon}_1 \varepsilon_2}{1 + \varepsilon_1 \bar{\varepsilon}_2} \cdot \begin{bmatrix} 1 & \frac{\varepsilon_1 + \varepsilon_2}{1 + \bar{\varepsilon}_1 \varepsilon_2} \\ \frac{\bar{\varepsilon}_1 + \bar{\varepsilon}_2}{1 + \varepsilon_1 \bar{\varepsilon}_2} & 1 \end{bmatrix} \quad (5.8)$$

Proof: Using the menhir representation (5.4), a composition of two boosts gives the following chain of equations – in which the matrices represent the Möbius action on the cromlech

$$\begin{bmatrix} 1 & \varepsilon_2 \\ \bar{\varepsilon}_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \varepsilon_1 \\ \bar{\varepsilon}_1 & 1 \end{bmatrix} \cdot z = \begin{bmatrix} 1 + \bar{\varepsilon}_1 \varepsilon_2 & \varepsilon_1 + \varepsilon_2 \\ \bar{\varepsilon}_1 + \bar{\varepsilon}_2 & 1 + \varepsilon_1 \bar{\varepsilon}_2 \end{bmatrix} \cdot z = \frac{1 + \bar{\varepsilon}_1 \varepsilon_2}{1 + \varepsilon_1 \bar{\varepsilon}_2} \cdot \begin{bmatrix} 1 & \frac{\varepsilon_1 + \varepsilon_2}{1 + \bar{\varepsilon}_1 \varepsilon_2} \\ \frac{\bar{\varepsilon}_1 + \bar{\varepsilon}_2}{1 + \varepsilon_1 \bar{\varepsilon}_2} & 1 \end{bmatrix} \cdot z$$

where we took advantage of (5.5): the numerator is extracted from the upper row of the matrix, and the denominator from the lower row. \square

Remark: Clearly (5.6) is in general a non-commutative product. However, if ε_1 and ε_2 are collinear, then they are both in the form $\varepsilon_k = \exp(i\varphi)e_k$ for some real $e_k, k = 1, 2$, which under substitution reduces equations (5.6) to (5.3).

Remark: The pairs (D, \boxplus) and (D, \oplus) are not groups but only loops. They have a neutral element, 0, and well-defined inverse, but are neither commutative nor associative. Yet we have the isomorphism

$$\mu(a) \oplus \mu(b) = \mu(a \boxplus b)$$

The results are summarized by the following commutative diagram or relations between the reversions, matrices, and celestial action of the Lorentz group:

$$\begin{array}{ccccccc} \text{matrices} & & \text{ALGEBRA} & & \text{GEOMETRY} & & \text{PHYSICS} \\ & & \text{Möbius action} & & \text{reversions} & & \text{relativity} \\ \frac{\mathbb{R}_+ \times \text{SU}(1, 1)}{S^1 \subset \mathbb{C}} \downarrow & \xrightarrow{\pi} & \frac{\text{PSU}(1, 1)}{S^1 \subset \mathbb{C}} \downarrow & \xleftrightarrow{1:1} & \frac{\text{Rev}_o(2)}{S^1 \subset \mathbb{R}^2} \downarrow & \xleftrightarrow{1:1} & \frac{\text{SO}_o(1, 2)}{\mathbb{R}_0^{1,2}} \downarrow \\ \downarrow \subset & & \downarrow \subset & & \downarrow \subset & & \downarrow \subset \\ \frac{\mathbb{R}_+ \times \text{SU}^\pm(1, 1)}{S^1 \subset \mathbb{C}} \downarrow & \xrightarrow{\pi} & \frac{\text{PSU}^\pm(1, 1)}{S^1 \subset \mathbb{C}} \downarrow & \xleftrightarrow{1:1} & \frac{\text{Rev}(2)}{S^1 \subset \mathbb{R}^2} \downarrow & \xleftrightarrow{1:1} & \frac{\text{SO}_o^\pm(1, 2)}{\mathbb{R}_0^{1,2}} \downarrow \end{array}$$

(5.9)

Group $\text{SO}_o^\pm(1, 2)$ is an extended Lorentz group that besides rotations (hyperbolic and elliptic) admits also reflections in space-like directions.

C. Beyond two dimensions – Quaternions \mathbb{H}

And how the One of Time, of Space the Three,
Might in the Chain of Symbols girdled be.
—William Rowan Hamilton

In order to consider more than two velocities, we need to move beyond $\mathbb{C} = \mathbb{R}^2$. The good news is that the above result may be extended to quaternions. This covers the case of $\mathbb{R}^{1,3}$, and thus includes the standard physical space-time. Since quaternions do not commute, the case needs additional care.

Remark on quaternions and relativity theory: A beginning student of science may – upon learning about quaternions – naïvely hold hopes that quaternions have something to do with the structure of space-time; after all the squares of the four basis elements of \mathbb{H} (i.e., $1, i, j, k$) have signs $(+, -, -, -)$, and that reminds the signature of the Minkowski space. As explained in the theory of Clifford algebras [11], it turns out that quaternions represent the standard rotations of the *Euclidean* spaces \mathbb{R}^3 and \mathbb{R}^4 . Hence it might be a source of surprise that we shall use quaternions to represent hyperbolic composition of velocities in the Minkowski space.

Basic facts: Quaternions form a division algebra $\mathbb{H} = \text{span}\{1, i, j, k\}$ with multiplication table

$$\begin{aligned} i^2 &= j^2 = k^2 = -1 \\ ij &= k = -ji \\ jk &= i = -kj \\ ki &= j = -ik \end{aligned}$$

Typical element of \mathbb{H} , a quaternion, is $q = a + bi + cj + dk$. The conjugation is denoted by either a star or a bar and is defined

$$\bar{q} = q^* = a - bi - cj - dk$$

Conjugation satisfies $(ab)^* = b^*a^*$. Norm squared is defined as $\|q\|^2 = qq^* = q^*q = a^2 + b^2 + c^2 + d^2$ and gives quaternions \mathbb{H} a Euclidean structure. It follows that $\|q\|^2 = \|\bar{q}\|^2$. Reciprocal (multiplicative inverse) is well-defined:

$$q^{-1} = \bar{q}/\|q\|^2.$$

The norm is multiplicative, that is it satisfies $|qp| = |q||p|$.

Convention on quaternion fractions: Quaternions are not commutative, thus there are two versions of division: left and right. We will understand quaternion *fractions* via the right inverses and shall assume the following notation:

$$\frac{p}{q} = pq^{-1}$$

Under such convention we have the following rules:

$$(i) \quad \frac{pa}{qa} = \frac{p}{q}, \quad (ii) \quad \frac{ap}{aq} = a\frac{p}{q}, \quad (iii) \quad \frac{p}{aq} = \frac{p}{q}a^{-1} \quad (5.10)$$

Proposition 5.4. *For any two 2-by-2 matrices with quaternion entries $M, N \in \text{Mat}(2, \mathbb{H})$ and $p, q, z \in \mathbb{H}$ the following holds:*

$$\begin{aligned} (i) \quad & (MN)z = M(Nz) \\ (ii) \quad & \frac{p}{q} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} pa & pb \\ qc & qd \end{bmatrix} \\ (iii) \quad & \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot z = azb^{-1} \end{aligned} \tag{5.11}$$

Proposition 5.5. *For any two quaternions $\varepsilon, \varphi \in \mathbb{H}$ the following left division may be replaced by right division as follows:*

$$(1 + \varphi\bar{\varepsilon})^{-1}(\varepsilon + \varphi) = (\varepsilon + \varphi)(1 + \bar{\varepsilon}\varphi)^{-1} \tag{5.12}$$

Proof: Since $|\varepsilon|^2 = \varepsilon\bar{\varepsilon} = \bar{\varepsilon}\varepsilon$ is a real number, it commutes with any quaternion, thus we have $\varepsilon\bar{\varepsilon}\varphi = \varphi\bar{\varepsilon}\varepsilon$. Therefore the following is true by expansion:

$$(\varepsilon + \varphi)(1 + \bar{\varepsilon}\varphi) = (1 + \varphi\bar{\varepsilon})(\varepsilon + \varphi),$$

which leads directly to (5.12). \square

This concludes our preliminary matter. Here is the main result:

Theorem 5.6. *A simple boost corresponds to quaternionic linear fractional action on $z \in S^3 \subset \mathbb{R}^4 \cong \mathbb{H}$ represented by the map:*

$$z \rightarrow z' = \frac{z + \varepsilon}{\bar{\varepsilon}z + 1} \equiv \begin{bmatrix} 1 & \varepsilon \\ \bar{\varepsilon} & 1 \end{bmatrix} \cdot z \tag{5.13}$$

($|z| = 1$) where ε is a quaternionic menhir for velocity v ,

$$v = \frac{2\varepsilon}{1 + |\varepsilon|^2} \tag{5.14}$$

Proof: Any non-real quaternion ε determines a complex plane $\text{span}\{1, \varepsilon\} = \mathbb{C}$ with $\text{Im } \varepsilon / |\text{Im } \varepsilon|$ acting as the “ $\sqrt{-1}$ ”. We consider two cases:

Case 1: Let $\varepsilon \in \mathbb{R}$ be real. For any z , z is either real – and (5.13) holds by Section A, or it is not – and then it reduces to the case of complex plane $\text{span}\{\varepsilon, z\} = \mathbb{R} \oplus [\varepsilon]$ of Section 5B.

Case 2: Assume that $\varepsilon \in \mathbb{H}$ is not real. Then the pair $\{1, \varepsilon\}$ spans a complex plane (with $n = \text{Im } \varepsilon / |\text{Im } \varepsilon|$ playing the role of the imaginary unit). The geometric mutual relations between x, x' , and ε is the same as between their uniformly rotated versions obtained by multiplying on the left by a unit quaternion $\bar{\varepsilon}/|\varepsilon|$. Thus we have a map:

$$\begin{aligned} z &\rightarrow \bar{\varepsilon}z/|\varepsilon| \\ z' &\rightarrow \bar{\varepsilon}z'/|\varepsilon| \\ \varepsilon &\rightarrow \bar{\varepsilon}\varepsilon/|\varepsilon| = |\varepsilon| \in \mathbb{R} \end{aligned}$$

Since they lie in the same plane with $|\varepsilon|$ being real, this boils down to the case 1 above. We can use the above result and write the map $z \rightarrow z'$:

$$z \rightarrow z' = \left(\frac{\bar{\varepsilon}}{|\varepsilon|} \right)^{-1} \frac{\frac{\bar{\varepsilon}z}{|\varepsilon|} + |\varepsilon|}{|\varepsilon| \frac{\bar{\varepsilon}z}{|\varepsilon|} + 1} = \left(\frac{\varepsilon}{|\varepsilon|} \right) \frac{\frac{\bar{\varepsilon}z}{|\varepsilon|} + |\varepsilon|}{\bar{\varepsilon}z + 1} = \frac{z + \varepsilon}{\bar{\varepsilon}z + 1},$$

where in the last equation we used property (5.10). \square

Theorem 5.7. [quaternion version of menhir calculus] *A composition of two consecutive boosts corresponding to menhirs ε and φ in $\mathbb{H} \cong \mathbb{R}^4$ are equivalent to a boost corresponding to menhir*

$$\varepsilon \boxplus \varphi = \frac{\varepsilon + \varphi}{1 + \bar{\varepsilon}\varphi} \quad (5.15)$$

followed by rotation that in terms of quaternions is represented by

$$q \rightarrow q' = (1 + \varphi\bar{\varepsilon}) q (1 + \bar{\varphi}\varepsilon)^{-1} \quad (5.16)$$

The master equation that conveys this information may be written as alternative splits of quaternion-valued matrices:

$$\begin{bmatrix} 1 & \varphi \\ \bar{\varphi} & 1 \end{bmatrix} \begin{bmatrix} 1 & \varepsilon \\ \bar{\varepsilon} & 1 \end{bmatrix} = \begin{bmatrix} 1 + \varphi\bar{\varepsilon} & 0 \\ 0 & 1 + \bar{\varphi}\varepsilon \end{bmatrix} \begin{bmatrix} 1 & \varepsilon \boxplus \varphi \\ (\varepsilon \boxplus \varphi)^* & 1 \end{bmatrix} \quad (5.17)$$

where $\varepsilon \boxplus \varphi$ is defined in (5.15).

Proof: We need to be careful due to non-commutativity of quaternions. Let us start with the composition of the two maps on the left-hand side of (5.17):

$$\begin{aligned} \begin{bmatrix} 1 & \varphi \\ \bar{\varphi} & 1 \end{bmatrix} \begin{bmatrix} 1 & \varepsilon \\ \bar{\varepsilon} & 1 \end{bmatrix} &= \begin{bmatrix} 1 + \varphi\bar{\varepsilon} & \varepsilon + \varphi \\ \bar{\varepsilon} + \bar{\varphi} & 1 + \bar{\varphi}\varepsilon \end{bmatrix} \\ &= \begin{bmatrix} 1 + \varphi\bar{\varepsilon} & 0 \\ 0 & 1 + \bar{\varphi}\varepsilon \end{bmatrix} \begin{bmatrix} 1 & (1 + \varphi\bar{\varepsilon})^{-1}(\varepsilon + \varphi) \\ (1 + \bar{\varphi}\varepsilon)^{-1}(\bar{\varepsilon} + \bar{\varphi}) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \varphi\bar{\varepsilon} & 0 \\ 0 & 1 + \bar{\varphi}\varepsilon \end{bmatrix} \begin{bmatrix} 1 & (\varepsilon + \varphi)(1 + \bar{\varepsilon}\varphi)^{-1} \\ ((\varepsilon + \varphi)(1 + \bar{\varepsilon}\varphi)^{-1})^* & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \varphi\bar{\varepsilon} & 0 \\ 0 & 1 + \bar{\varphi}\varepsilon \end{bmatrix} \begin{bmatrix} 1 & \frac{\varepsilon + \varphi}{1 + \bar{\varepsilon}\varphi} \\ \left(\frac{\varepsilon + \varphi}{1 + \bar{\varepsilon}\varphi} \right)^* & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \varphi\bar{\varepsilon} & 0 \\ 0 & 1 + \bar{\varphi}\varepsilon \end{bmatrix} \begin{bmatrix} 1 & \varepsilon \boxplus \varphi \\ (\varepsilon \boxplus \varphi)^* & 1 \end{bmatrix} \end{aligned}$$

where we used (5.10). This establishes (5.17). The second matrix on the right of (5.17) represents a boost. To interpret the first as a rotation in \mathbb{H} , we need $|1 + \varphi\bar{\varepsilon}| = |1 + \bar{\varphi}\varepsilon|$, which is easy to check. \square

The result may be presented in a matrix-free form:

$$z'' = (1 + \varphi\bar{\varepsilon}) \frac{z + \frac{\varepsilon + \varphi}{1 + \bar{\varepsilon}\varphi}}{\left(\frac{\varepsilon + \varphi}{1 + \bar{\varepsilon}\varphi} \right)^* z + 1} (1 + \bar{\varphi}\varepsilon)^{-1}, \quad (5.18)$$

The last theorem encompasses all previous lower-dimensional cases. We may state it as a general mathematical fact: Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be a division algebra. Special 2×2 matrices are denoted:

$$M(\varphi) = \begin{bmatrix} 1 & \varphi \\ \bar{\varphi} & 1 \end{bmatrix}, \quad R(\alpha, \beta) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}. \quad (5.19)$$

for any $\alpha, \beta, \varphi \in \mathbb{F}$, with $|\alpha| = |\beta|$. Then the following two factorizations are equivalent:

$$M(\varphi)M(\varepsilon) = R(1 + \varphi\bar{\varepsilon}, 1 + \bar{\varphi}\varepsilon)M(\varepsilon \boxplus \varphi).$$

It is only a matter of interpretation that we associate with these objects the following meanings:

- φ = “Poincaré square root of velocity”: $v = 2\varphi/(1 + |\varphi|^2)$, and $M(\varphi)$ = matrix of Möbius action on the sphere of unit numbers $x \in \mathbb{F} : \|x\|^2 = 1$.
- $R(\alpha, \beta)$ = matrix of rotation of the unit sphere in \mathbb{F} through the “sandwich” action $x \rightarrow \alpha x \beta^{-1}$.

The matrices generated a symplectic matrix group

$$\mathrm{Sp}(1, 1) = \left\{ A = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \mid a, b \in \mathbb{H} \right\}$$

We actually utilize scaled matrices with the property $a\bar{a} + b\bar{b} < 1$. The scaling is convenient in defining the correspondence (5.13) but is irrelevant since ultimately we deal with the projective group $\mathrm{PSp}(1, 1)$.

The whole situation is summarized below in (5.20).

$$\begin{array}{ccccc}
 \text{PHYSICS} & & & & \text{ALGEBRA} \\
 \text{relativity} & & & & \text{Möbius action} \\
 \\
 \frac{\mathrm{SO}_0(1, 4)}{\mathbb{R}^{1,4}} \downarrow & \xleftarrow{2:1} & & & \frac{\mathrm{Sp}(1, 1)}{\mathbb{H}^2} \downarrow \\
 \\
 \downarrow & & \text{GEOMETRY} & & \downarrow \\
 & & \text{reversions} & & \\
 \frac{\mathrm{PSO}_0(1, 4)}{\mathbb{PR}_0^{1,4} \cong S^3} \downarrow & \xleftarrow{1:1} & \frac{\mathrm{Rev}(4)}{S^3 \subset \mathbb{R}^3} \downarrow & \xleftarrow{1:1} & \frac{\mathrm{PSp}(1, 1)}{S^3 \subset \mathbb{H}} \downarrow \\
 \text{CELESTIAL} & & \text{CROMLECH} & & \text{numbers} \\
 \text{SPHERE} & & & &
 \end{array} \quad (5.20)$$

D. Back to reality: three-dimensional space and quaternions

In order to describe the standard three-dimensional case, we can simply reduce the above result to a three-dimensional subspace of \mathbb{H} , for instance the imaginary part. We shall identify vectors of \mathbb{R}^3 with $\text{Im } \mathbb{H} = \text{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in the obvious way. It is easy to check that the transformation of type (5.19) preserves the imaginary part of the unit sphere in \mathbb{H} , i.e., the unit sphere in $\text{Im } \mathbb{H}$:

$$z, \varepsilon \in \text{Im } \mathbb{H} \cap S^3 \quad \Rightarrow \quad \frac{z + \varepsilon}{1 + \bar{\varepsilon}z} \in \text{Im } \mathbb{H} \cap S^3.$$

(simple calculational verification). Note, that the constraint to $\text{Im } \mathbb{H}$ implies that $\bar{\varepsilon} = -\varepsilon$, and therefore, in the matrix representation of the rotational part an interesting thing happens: $\bar{\varphi}\varepsilon = \varphi\bar{\varepsilon} = -\varphi\varepsilon$. This simplifies the transformation (5.18) to

$$z'' = (1 - \varphi\varepsilon) \frac{z + \frac{\varepsilon + \varphi}{1 - \varepsilon\varphi}}{\left(\frac{\varepsilon + \varphi}{1 - \varepsilon\varphi}\right)^* z + 1} (1 - \varphi\varepsilon)^{-1},$$

The exterior part of the transformation

$$z' = q z' q^{-1} \quad \text{where} \quad q = 1 - \varphi\varepsilon,$$

may readily be recognized as the standard Hamilton's trick to represent rotations of \mathbb{R}^3 by quaternions via adjoint action: $z \rightarrow qzq^{-1}$. The composition of two boosts splits according to:

$$B_\varphi B_\varepsilon = R(1 - \varphi\varepsilon) B(\varepsilon \boxplus \varphi)$$

where:

$$\text{Axis of rotation: } A = \text{Im}(1 - \varphi\varepsilon) = \text{Im } \varphi\varepsilon$$

$$\text{Angle of rotation: } \theta = 2 \arccos(\text{Re}(1 - \varphi\varepsilon))/|1 - \varphi\varepsilon|$$

E. Higher dimensions — Clifford algebra

Minkowski spaces beyond the standard 1+3 case are interesting for applications in physics (Kaluza-Klein model or string theories are examples). To extend our model to such cases the fields \mathbb{R} , \mathbb{C} , and \mathbb{H} must be replaced by Clifford algebras. Recall that for a given Euclidean space (\mathbf{E}, g) with metric g , the universal Clifford algebra $\text{Cliff}(\mathbf{E})$ is a $2^{(\dim \mathbf{E})}$ – dimensional space that can be identified with the Grassmann algebra $\wedge \mathbf{E}$ with product that for two vectors $v, w \in \mathbf{E} \subset \text{Cliff}(\mathbf{E})$ is

$$\mathbf{vw} = -g(\mathbf{v}, \mathbf{w}) + \mathbf{v} \wedge \mathbf{w}$$

In particular $\mathbf{v}^2 = \mathbf{vv} = -\|\mathbf{v}\|^2 \equiv v^2$. (We shall denote the norm with a non-bold letters: $v = \|\mathbf{v}\|$.) Define conjugation in the Clifford algebra as $(\mathbf{ab})^* = \mathbf{b}^* \mathbf{a}^*$ for arbitrary elements of $\text{Cliff}(\mathbf{E})$, and $\mathbf{v}^* = -\mathbf{v}$ for $\mathbf{v} \in E$. In particular, for any orthonormal basis in \mathbf{E} we have

$$(\mathbf{e}_{i_1} \mathbf{e}_{i_2} \dots \mathbf{e}_{i_k})^* = (-\mathbf{e}_{i_k}) \dots (-\mathbf{e}_{i_2})(-\mathbf{e}_{i_1})$$

We shall use the same convention for “fractions” in $\text{Cliff}(\mathbf{E})$ as in the case of quaternions:

$$\frac{\mathbf{p}}{\mathbf{q}} = \mathbf{pq}^{-1}$$

for invertible elements $\mathbf{q} \in \text{Cliff}(\mathbf{E})$. Analogously to the previous models, let us define :

$$\begin{array}{ll} \text{cromlech} & K = \{\mathbf{f} \in \mathbf{E} \mid |\mathbf{f}|^2 = 1\} \\ \text{menhir disk} & D = \{\mathbf{z} \in \mathbf{E} \mid |\mathbf{z}|^2 < 1\}. \end{array}$$

These elements are now understood in the context of the Clifford algebra.

Proposition 5.8. *1. The following Möbius action preserves sphere K*

$$\mathbf{z} \rightarrow \mathbf{z}' = \begin{bmatrix} 1 & \mathbf{f} \\ -\mathbf{f} & 1 \end{bmatrix} \cdot \mathbf{z} = \frac{\mathbf{z} + \mathbf{f}}{-\mathbf{f}\mathbf{z} + 1} = (\mathbf{z} + \mathbf{f})(1 - \mathbf{f}\mathbf{z})^{-1} \in K$$

for any $\mathbf{f} \in D$ and $\mathbf{z} \in K$.

Note that the denominator contains in general a bivector (a component of $\mathbf{f}\mathbf{z}$). Yet the “rationalization” of the denominator leads to the stated result. Define two types of matrices

$$M(\mathbf{f}) = \begin{bmatrix} 1 & \mathbf{f} \\ -\mathbf{f} & 1 \end{bmatrix}, \quad R(b) = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

By similar arguments as in the previous section we have a general result.

Theorem 5.9. *The composition of two boosts admits an alternative decomposition as follows:*

$$M(\mathbf{e})M(\mathbf{f}) = R(1 - \mathbf{f}\mathbf{e})M(\mathbf{e} \boxplus \mathbf{f})$$

for any $\mathbf{e}, \mathbf{f} \in D$. In explicit terms,

$$\begin{bmatrix} 1 & \mathbf{e} \\ -\mathbf{e} & 1 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{f} \\ -\mathbf{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \mathbf{f}\mathbf{e} & 0 \\ 0 & 1 - \mathbf{f}\mathbf{e} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{e} \boxplus \mathbf{f} \\ -\mathbf{e} \boxplus \mathbf{f} & 1 \end{bmatrix}$$

where $\mathbf{e} \boxplus \mathbf{f}$ is defined

$$\mathbf{e} \boxplus \mathbf{f} = \frac{\mathbf{e} + \mathbf{f}}{1 - \mathbf{e}\mathbf{f}}$$

The proofs of the above claims may be based on the fact that any two vectors \mathbf{e} and \mathbf{f} . Introduce an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ in $\text{span}\{\mathbf{e}, \mathbf{f}\}$. The space $\text{span}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\}$ behaves like quaternions and we may thus rewrite the formulas obtained in this section, part E. One may also simply multiply matrices on both sides and use the fact that

$$(1 - \mathbf{f}\mathbf{e})(\mathbf{e} + \mathbf{f}) = (\mathbf{e} + \mathbf{f})(1 - \mathbf{e}\mathbf{f})$$

because $(\mathbf{f}\mathbf{e})\mathbf{e} = \mathbf{f}(\mathbf{e}\mathbf{e}) = (\mathbf{e}\mathbf{e})\mathbf{f} = \mathbf{e}(\mathbf{e}\mathbf{f})$, and similarly for \mathbf{e} .

All this establishes the isomorphism

$$\text{SO}_0(1, n) \cong \text{gen}\{M(\mathbf{f}) \mid \mathbf{f} \in D\}$$

The matrices are agreed with the \mathbb{Z}_2 grading of the Clifford algebra: in the sense that the diagonal entries are odd elements of $\text{Cliff}(\mathbf{E})$ and the off-diagonal – the even elements.

Remark: Quite interestingly, we may model a similar action by using elements of

$$\mathbb{R} \oplus \mathbf{E} \subset \text{Cliff}(\mathbf{E}, g)$$

whose typical element is a formal sum of a scalar and a vector, $a = \alpha + v$. with conjugation of a in \dot{E} defined as $a^* \equiv \bar{a} = \alpha - v$. This however will be explored elsewhere.

6. Geometric construction of the composition of velocities

Here explore the geometric side of the “menhir calculus” in terms of reversions, see Section 4. The dimension is arbitrary but the figures are drawn for the 2-dimensional case. Recall our notation and basic facts:

D = unit disk of dimension n

$K = \partial D$ = unit sphere of dimension $n - 1$

Reversion through a point $p \in D$ is a map denoted by bold $\mathbf{p} : K \rightarrow K$. Reversions may be composed, as in Figure 2.1, and group $\text{Rev}(K) = \text{gen } \{\mathbf{p} \mid p \in D\}$.

Definition 6.1. The menhir $e \in D$ of velocity $v \in D$ is defined by

$$\mathbf{e}\mathbf{v}\mathbf{e} = \text{id} \quad \text{or equivalently} \quad \mathbf{o}\mathbf{e} = \mathbf{e}\mathbf{v}.$$

The physical meanings: The unit circle K , cromlech, represents horizon. A boost by velocity v causes aberration of star positions, namely a star originally visible at $A \in K$ will become visible at

$$A' = A\mathbf{o}\mathbf{e}$$

We start with a Lemma on butterfly, recalled here from [4]:

Lemma 6.2. Suppose points $p, q, r, s \in D$ are collinear. Then if $A\mathbf{p}\mathbf{q}\mathbf{r}\mathbf{s} = A$ for some $A \in K$ then $\mathbf{p}\mathbf{q}\mathbf{r}\mathbf{s} = \text{id}$ (see Figure 6.1)

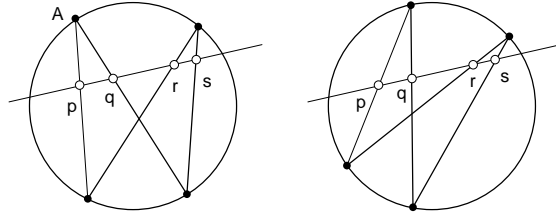


FIGURE 6.1: The butterfly porism theorem

Corollary 6.3. For any point a' on the line (a, b) there exists a point b' on this line such that $A\mathbf{a}\mathbf{b} = A\mathbf{a}'\mathbf{b}'$ for any $A \in K$. In particular

$$\mathbf{o}\mathbf{e} = (-\mathbf{e}')\mathbf{o}$$

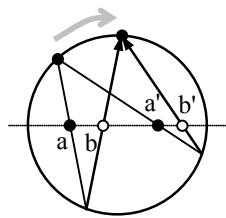


FIGURE 6.2: $\mathbf{a}\mathbf{b} = \mathbf{a}'\mathbf{b}'$

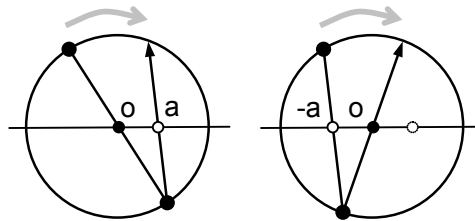


FIGURE 6.3: $\mathbf{o}\mathbf{a} = (-\mathbf{a})\mathbf{o}$

Corollary 6.4. *Composition of two boosts related to menhirs e and f may be represented by a pair of menhirs, namely $(-e)f$.*

Proof: Readily follows from the above Corollary: $(oe)(of) = ((-e)o)(of) = (-e)f$. See also Figure 6.6 \square

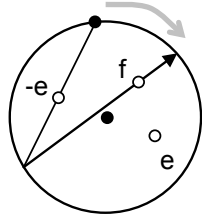


FIGURE 6.4: Composition of two boosts in the menhir representation

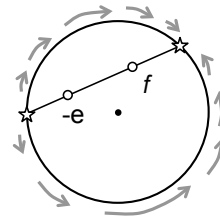


FIGURE 6.5: Nonuniform shift of stars under a composition of two boosts

Composition of two boosts is not a single boost. In particular, notice that the stable points are not antipodal (see Figure 6.5). But it is a composition of a single boost and a rotation. In order to find it, we need first to “subtract” the rotational part.

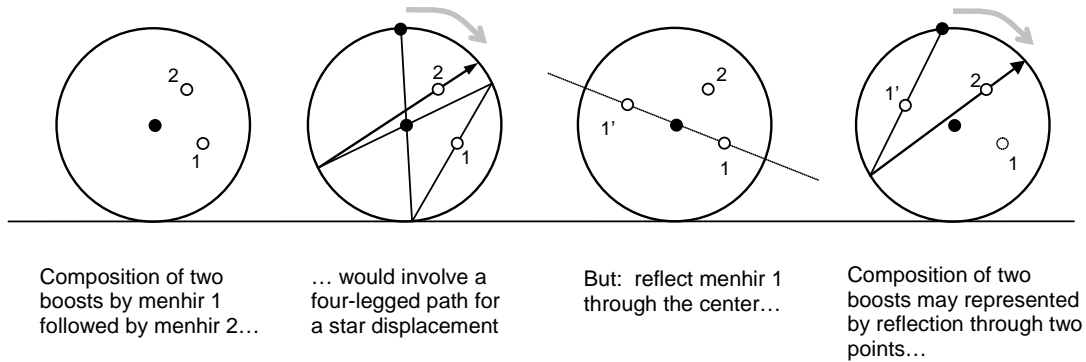


FIGURE 6.6: Proof. Points are labeled by numbers

Proposition 6.5. *The rotation cased by a composition of boosts $B(e)B(f)$ (first e and then f) can be constructed as shown in Figure 6.7. The angle of rotation is (AoB) .*

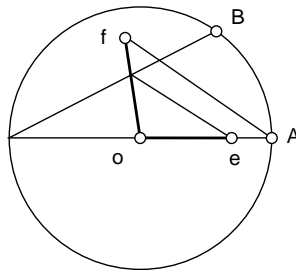


FIGURE 6.7: The rotational component of $B(f)B(e)$ is such that A goes to B .

Here are the steps of the construction:

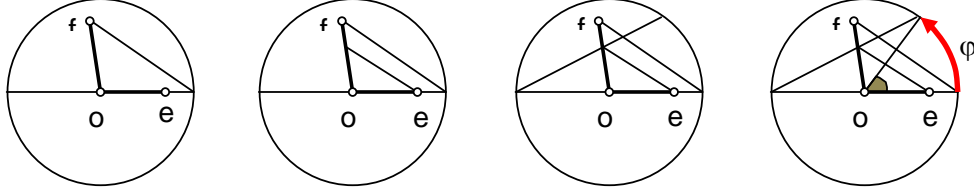


FIGURE 6.8: Steps of the construction of the angle.

Proof: Analyse Equation 5.7. The angle of rotation is given by

$$\rho = e^{i\theta} = \frac{1 + \varepsilon_2 \bar{\varepsilon}_1}{1 + \bar{\varepsilon}_2 \varepsilon_1} = \frac{(1 + \varepsilon_2 \bar{\varepsilon}_1)^2}{|1 + \bar{\varepsilon}_2 \varepsilon_1|^2},$$

which implies that the angle θ is twice the $\text{Arg}(1 + \varepsilon_2 \bar{\varepsilon}_1)$. Hence the construction. \square

In order to construct the velocity $v \oplus w$, or rather the corresponding menhir $e \boxplus f$, construct first A and its image B as above. Note that their antipodals A' and B' also differ by the same rotation angle. Do the following construction for the pair (A, B) and then repeat for (A', B') to get segments α and β :

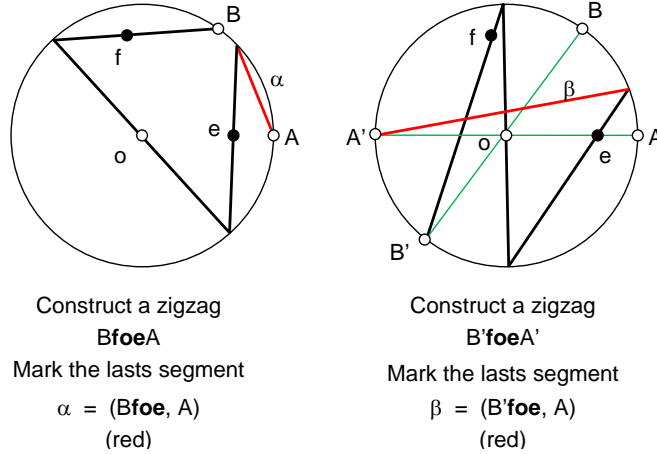


FIGURE 6.9: $e \boxplus f = \alpha \cap \beta$.

Proposition 6.6. *The intersection of segments α and β is $e \boxplus f$, the menhir of the velocity $v \oplus w$. That is:*

$$e \boxplus f = (Bfoe, A) \cap (B'foe, A')$$

7. Appendix

7.1. Dictionary for prehistoric megalithic objects

Cromlech is a Welsh and Brytonic word for megalithic structures. Some authors limit its usage to dolmens, but we follow the French custom to apply it to megalithic stone circles. From *cromm+llech*= *bend stone*.

Menhir is a large upright single stone that is typically a part of a larger structure system (but not a part of a single architectural construction). Breton *men+hir* = *stone-long*.

7.2. Associativity and the relativistic composition of velocities.

We may now probe the feature of non-associativity of the “addition of velocities.” We have in general:

- (i) $v \oplus (-v) = 0$
- (ii) $v \oplus w \neq w \oplus v$
- (iii) $(v \oplus w) \oplus u \neq v \oplus (w \oplus u)$

This makes the the unit disk a loop. However, the composition of the boosts understood in terms of the matrices (or reversion) is associative (although non-commutative) and forms a group. For instance in the quaternion case:

- (i) $M_\varphi M_\varepsilon \neq M_\varepsilon M_\varphi$
- (ii) $(M_\varepsilon M_\varphi) M_\delta = M_\varepsilon (M_\varphi M_\delta)$

where we can consider the quaternion version

$$M_\varepsilon = \begin{bmatrix} 1 & \varepsilon \\ \bar{\varepsilon} & 1 \end{bmatrix}$$

Denote

$$R(\varepsilon, \varphi) = \begin{bmatrix} 1 + \varphi \bar{\varepsilon} & 0 \\ 0 & 1 + \bar{\varphi} \varepsilon \end{bmatrix}$$

The composition of three boosts depends on the bracketing:

$$\begin{aligned} (M(a) M(b)) M(c) &= M(a \boxplus b) R(a, b) M(c) \\ &= M(a \boxplus b) M(R^{-1}(a, b) c) R(a, b) \\ &= M((a \boxplus b) \boxplus R^{-1}(a, b) c) R(a \boxplus b, R^{-1}(a, b) c) R(a, b) \end{aligned}$$

versus

$$\begin{aligned} M(a) (M(b) M(c)) &= M(a) M(b \boxplus c) R(b, c) \\ &= M(a \boxplus (b \boxplus c)) R(a, b \boxplus c) R(b, c) \end{aligned}$$

The seeming peculiarity of “non-associativity of relativistic addition of velocities” results as a careless extension of the intuition based in the Galilean-Newtonian physics.

7.2. “Adding velocities?” – misunderstandings and clarifications

It is sometimes stated that the essence of the theory of relativity lies in the Lorentz group. A more accurate view seems that the heart the theory of relativity lies in geometry. Minkowski space is a pair (\mathbf{M}, g_M) where \mathbf{M} is a linear space with an inner product of signature $(1, n)$ (one plus and n minuses). The Lorentz group emerges as the symmetry group of this product. The structure of Minkowski space – quite like that of Euclidean space – is entirely determined by its “unit sphere”, the hyperboloid of space-like vectors $\|\mathbf{v}\|^2 = -1$, and two component hyperboloid made of time-like vectors $\|\mathbf{v}\|^2 = +1$ (one “past” and one “future”).

One must distinguish between these three concepts: that of **velocity**, that of **observer**, and that of a **lab**.

A. An observer at p is tantamount to a unit time-like future oriented vector, \mathbf{T} . Such a vector determines a split of the space into a Cartesian product

$$M = \text{span}\{\mathbf{T}\} \times \mathbf{T}^\perp = \text{“space”} \times \text{“time”}$$

of a 1-dimensional “time axis” $\text{span}\{\mathbf{T}\}$ and 1-codimensional subspace, the “instantaneous space” $\mathbf{T}^\perp = \{\mathbf{u} \in M \mid \mathbf{u} \perp \mathbf{T}\}$ (all vectors perpendicular to \mathbf{T} in the sense of metric g_M). Geometrically, the space-like subspace is determined by the tangent to the sphere at \mathbf{T} , see Figure 7.1 for the case $\mathbb{R}^{1,1}$.

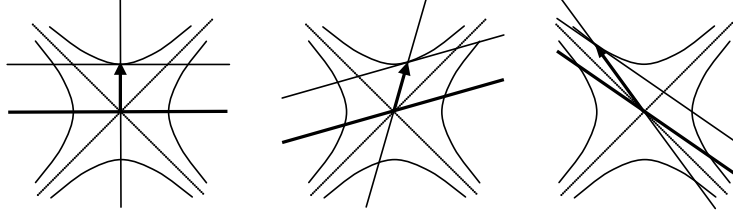


FIGURE 7.1: Various observers come automatically with a split of space-time into “space” times “time”.

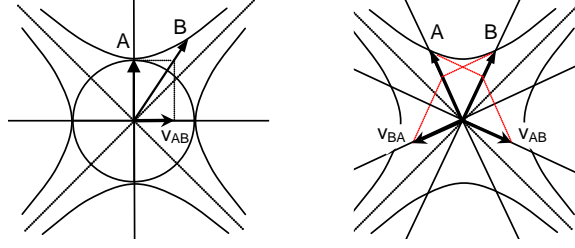
Thus the points of the upper hyperboloid parameterize observers.¹

B. Velocity is a measure of how one observer relates to another observer. The expression “observer B has velocity \mathbf{v}_{AB} with respect to observer A ” should be represented as a space-like vector \mathbf{v}_{AB} in the A -space, see Figure 7.2, left. There is also a vector \mathbf{v}_{BA} of the velocity of observer A with respect to B . What is not always clearly realized is that \mathbf{v}_{AB} and \mathbf{v}_{BA} are not negatives of each other, as one may easily see from the Figure 7.2, right:

$$\mathbf{v}_{AB} + \mathbf{v}_{BA} \neq 0.$$

Such a sum is actually a time-like past-oriented vector. So, how come we tend to think naïvely that a boost followed by the “inverse boost” corresponds to simple expression $\mathbf{v} + (-\mathbf{v}) = 0$, obviously wrong in the light of the above? In order to make sense of “addition of velocities”, we need another concept – that of a “lab”.

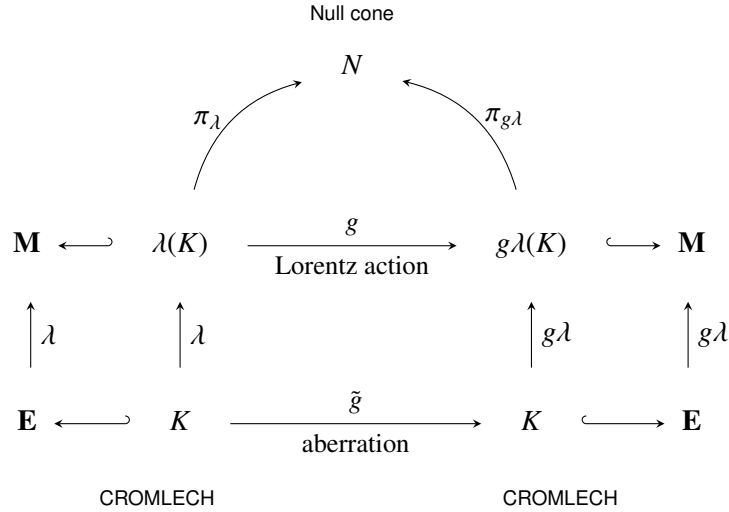
¹Observer as $(1,1)$ -tensor field in Lorentz manifold see [5].

FIGURE 7.2: Problems with misconceptualized equation " $v + (-v) = 0$ ".

C. A lab represents the space as we experience it. We can model it as an n -dimensional Euclidean space E and its space-time configuration as a linear map

$$\lambda : \mathbf{E} \rightarrow \mathbf{M} \cong \mathbb{R}^{1,n}$$

such that the inner product induced from the Minkowski space \mathbf{M} agrees with that of \mathbf{E} . Such a map defines \mathbf{T}_λ , a unit future-oriented vector perpendicular to the embedded space, $\mathbf{T}_\lambda \perp \lambda(\mathbf{E})$. The idea is that any Lorentz transformation that reorients \mathbf{E} in \mathbf{M} can be detected via pull-back map of features in \mathbf{M} back to \mathbf{E} .

FIGURE 7.3: Pullback of Lorentz transformation g to cromlech K

In particular, every λ establishes a 1-1 correspondence between the celestial sphere $N = \mathbb{P}\mathbb{R}_0^{1,n}$ (projective null-space) and the unit sphere K in \mathbf{E} (cromlech), namely as the map $\lambda^{-1} \circ \pi_\lambda$, where the projection π is described in Section 3. Consequently, any Lorentz map $g \in \Lambda$ of \mathbf{M} can be pulled back to \mathbf{E} as a conformal diffeomorphism \tilde{g} of the unit sphere K via the commutativity of the diagram in Figure 7.3. In particular, any vector of velocity represented as $\mathbf{v} \in \mathbf{E}$ determines a hyperbolic rotation (boost) in the plane $\mathbf{T}_\lambda \wedge \lambda(\mathbf{v})$, namely the group element

$$B_{\lambda, \mathbf{v}} = \left(\frac{1 + v}{1 - v} \right)^{\frac{\mathbf{T}_\lambda \wedge \lambda(\mathbf{v})}{4v}}$$

which, from the perspective of \mathbf{E} , we simply denote as B_v . This pull-back of the action of the Lorentz group to the cromlech allows us for the geometric and algebraic constructions described in the present paper.

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